Exercise 4D

1 **a** $\int \frac{1}{5-4x-x^2}\,\mathrm{d}x$

$5-4x-x^2 = 5-\left(4x+x^2\right)$

$= 9-\left(x+2\right)^2$

Let $x+2=3\sin\theta \Rightarrow \mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$

$$\int \frac{1}{\sqrt{5-4x-x^2}}\,\mathrm{d}x = \int \frac{3}{\sqrt{9-9\sin^2\theta}}\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{1}{\sqrt{\cos^2\theta}}\cos\theta\,\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta$$

$$= \theta + c$$

$3\sin\theta = x+2 \Rightarrow \theta = \arcsin\left(\frac{x+2}{3}\right)$

$$\int \frac{1}{\sqrt{5-4x-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x+2}{3}\right)+c$$

b $\int \frac{1}{\sqrt{x^2-4x-12}}\,\mathrm{d}x$

$x^2-4x-12 = \left(x-2\right)^2-16$

$$\int \frac{1}{\sqrt{x^2-4x-12}}\,\mathrm{d}x = \int \frac{1}{\sqrt{\left(x-2\right)^2-4^2}}\,\mathrm{d}x$$

Let $u = x-2 \Rightarrow \mathrm{d}u = \mathrm{d}x$

$$\int \frac{1}{\sqrt{x^2-4x-12}}\,\mathrm{d}x = \int \frac{1}{\sqrt{u^2-4^2}}\,\mathrm{d}u$$

$$= \operatorname{arcosh}\left(\frac{u}{4}\right)+c$$

$$= \operatorname{arcosh}\left(\frac{x-2}{4}\right)+c$$

1 c $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx$

$x^2+6x+10=(x+3)^2+1$

$\int \frac{1}{\sqrt{x^2+6x+10}}\,dx=\int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$

Let $u=x+3 \Rightarrow du=dx$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+6x+10}}\,dx &= \int \frac{1}{\sqrt{u^2+1}}\,du \\ &= \text{arsinh}\,u+c \\ &= \text{arsinh}\,(x+3)+c\end{aligned}$$

d $\int \frac{1}{\sqrt{x(x-2)}}\,dx$

$x^2-2x=(x-1)^2-1$

$\int \frac{1}{\sqrt{x(x-2)}}\,dx=\int \frac{1}{\sqrt{(x-1)^2-1}}\,dx$

Let $u=x-1 \Rightarrow du=dx$

$$\begin{aligned}\int \frac{1}{\sqrt{x(x-2)}}\,dx &= \int \frac{1}{\sqrt{u^2-1}}\,dx \\ &= \text{arcosh}\,u+c \\ &= \text{arcosh}\,(x-1)+c\end{aligned}$$

1 e $\int \frac{1}{2x^2+4x+7}\,dx$

$$2x^2+4x+7 = 2\left(x^2+2x+\frac{7}{2}\right)$$
$$= 2\left((x+1)^2-1+\frac{7}{2}\right)$$
$$= 2\left((x+1)^2+\frac{5}{2}\right)$$

$$\int \frac{1}{2x^2+4x+7}\,dx = \int \frac{1}{2\left((x+1)^2+\frac{5}{2}\right)}\,dx$$
$$= \frac{1}{2}\int \frac{1}{(x+1)^2+\frac{5}{2}}\,dx$$

Let $x+1=\sqrt{\frac{5}{2}}\tan\theta \Rightarrow dx = \sqrt{\frac{5}{2}}\sec^2\theta\,d\theta$

$$\int \frac{1}{2x^2+4x+7}\,dx = \sqrt{\frac{5}{2}}\times\frac{1}{2}\int \frac{1}{\frac{5}{2}\tan^2\theta+\frac{5}{2}}\sec^2\theta\,d\theta$$
$$= \sqrt{\frac{5}{2}}\times\frac{1}{2}\int \frac{1}{\frac{5}{2}\left(\tan^2\theta+1\right)}\sec^2\theta\,d\theta$$
$$= \frac{2}{5}\times\sqrt{\frac{5}{2}}\times\frac{1}{2}\int \frac{1}{\tan^2\theta+1}\sec^2\theta\,d\theta$$
$$= \frac{1}{5}\sqrt{\frac{5}{2}}\int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta$$
$$= \frac{1}{\sqrt{10}}\int d\theta$$
$$= \frac{1}{\sqrt{10}}\theta + c$$

$$x+1=\sqrt{\frac{5}{2}}\tan\theta \Rightarrow \theta = \arctan\left(\sqrt{\frac{2}{5}}\times(x+1)\right)$$

$$\int \frac{1}{2x^2+4x+7}\,dx = \frac{1}{\sqrt{10}}\arctan\left(\sqrt{\frac{2}{5}}\times(x+1)\right)+c$$

1 f $\int \frac{1}{\sqrt{-4x^2-12x}}\,dx$

$-4x^2-12x = 9-(2x+3)^2$

$\int \frac{1}{\sqrt{-4x^2-12x}}\,dx = \int \frac{1}{\sqrt{9-(2x+3)^2}}\,dx$

Let $u = 2x+3 \Rightarrow dx = \frac{1}{2}du$

$$\begin{aligned}\int \frac{1}{\sqrt{-4x^2-12x}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{9-u^2}}\,du \\ &= \frac{1}{2}\int \frac{1}{\sqrt{3^2-u^2}}\,du \\ &= \frac{1}{2}\arcsin\left(\frac{u}{3}\right)+c \\ &= \frac{1}{2}\arcsin\left(\frac{2x+3}{3}\right)+c\end{aligned}$$

g $\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx$

$$\begin{aligned}14-12x-2x^2 &= 2(7-6x-x^2) \\ &= 2(7-(6x+x^2)) \\ &= 2(7+9-(3+x)^2) \\ &= 2(16-(3+x)^2)\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx &= \int \frac{1}{\sqrt{2(16-(3+x)^2)}}\,dx \\ &= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{4^2-(3+x)^2}}\,dx\end{aligned}$$

Let $u = x+3 \Rightarrow du = dx$

$$\begin{aligned}\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx &= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{4^2-u^2}}\,du \\ &= \frac{1}{\sqrt{2}}\arcsin\left(\frac{u}{4}\right)+c \\ &= \frac{1}{\sqrt{2}}\arcsin\left(\frac{x+3}{4}\right)+c\end{aligned}$$

1 h $\int \frac{1}{\sqrt{9x^2 - 8x + 1}} \, dx$

$$9x^2 - 8x + 1 = 9\left(x^2 - \frac{8}{9}x + \frac{1}{9}\right)$$

$$= 9\left(\left(x - \frac{4}{9}\right)^2 - \frac{16}{81} + \frac{1}{9}\right)$$

$$= 9\left(\left(x - \frac{4}{9}\right)^2 - \frac{7}{81}\right)$$

$$\int \frac{1}{\sqrt{9x^2 - 8x + 1}} \, dx = \int \frac{1}{\sqrt{9\left(\left(x - \frac{4}{9}\right)^2 - \frac{7}{81}\right)}} \, dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(x - \frac{4}{9}\right)^2 - \frac{7}{81}}} \, dx$$

Let $u = x - \frac{4}{9} \Rightarrow du = dx$

$$\int \frac{1}{\sqrt{9x^2 - 8x + 1}} \, dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2 - \frac{\sqrt{7}}{9}}} \, du$$

$$= \frac{1}{3}\text{arcosh}\left(\frac{9}{\sqrt{7}}u\right) + c$$

$$= \frac{1}{3}\text{arcosh}\left(\frac{9}{\sqrt{7}}\left(x - \frac{4}{9}\right)\right) + c$$

2 a $\int \frac{1}{\sqrt{4x^2 - 12x + 10}} \, dx$

$$4x^2 - 12x + 10 = (2x - 3)^2 - 9 + 10$$

$$= (2x - 3)^2 + 1$$

$$\int \frac{1}{\sqrt{4x^2 - 12x + 10}} \, dx = \int \frac{1}{\sqrt{(2x - 3)^2 + 1}} \, dx$$

Let $u = 2x - 3 \Rightarrow dx = \frac{1}{2}du$

$$\int \frac{1}{\sqrt{4x^2 - 12x + 10}} \, dx = \frac{1}{2}\int \frac{1}{\sqrt{u^2 + 1}} \, du$$

$$= \frac{1}{2}\text{arsinh}\, u + c$$

$$= \frac{1}{2}\text{arsinh}\,(2x - 3) + c$$

2 b $\displaystyle\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx$

$4x^2-12x+4=(2x-3)^2-9+4$

$=(2x-3)^2-5$

$$\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx=\int \frac{1}{\sqrt{(2x-3)^2-5}}\,dx$$

Let $u=2x-3 \Rightarrow dx=\frac{1}{2}du$

$$\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx=\int \frac{1}{\sqrt{u^2-\left(\sqrt{5}\right)^2}}\,dx$$

$$=\frac{1}{2}\text{arcosh}\left(\frac{u}{\sqrt{5}}\right)+c$$

$$=\frac{1}{2}\text{arcosh}\left(\frac{2x-3}{\sqrt{5}}\right)+c$$

3 a $\int_1^2 \frac{1}{\sqrt{x^2+2x+5}}\,dx$

$x^2+2x+5=(x+1)^2-1+5$

$=(x+1)^2+4$

$\int_1^2 \frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_1^2 \frac{1}{\sqrt{(x+1)^2+4}}\,dx$

Let $u=x+1 \Rightarrow du=dx$

When $x=1$, $u=2$

When $x=2$, $u=3$

$\int_1^2 \frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_2^3 \frac{1}{\sqrt{u^2+2^2}}\,du$

$=\left[\text{arsinh}\left(\frac{u}{2}\right)\right]_2^3$

$=\text{arsinh}\left(\frac{3}{2}\right)-\text{arsinh}(1)$

Since $\text{arsinh}x=\ln\left(x+\sqrt{x^2+1}\right)$

$\int_1^2 \frac{1}{\sqrt{x^2+2x+5}}\,dx=\ln\left(\frac{3}{2}+\sqrt{\left(\frac{3}{2}\right)^2+1}\right)-\ln\left(1+\sqrt{1^2+1}\right)$

$=\ln\left(\frac{3}{2}+\sqrt{\frac{13}{4}}\right)-\ln\left(1+\sqrt{2}\right)$

$=\ln\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)-\ln\left(1+\sqrt{2}\right)$

$=\ln\left(\frac{3+\sqrt{13}}{2}\right)-\ln\left(1+\sqrt{2}\right)$

$=\ln\left(\frac{3+\sqrt{13}}{2\left(1+\sqrt{2}\right)}\right)$

$=0.313$ (3 s.f.)

3 b $\displaystyle\int_1^3 \frac{1}{x^2+x+1}\,dx$

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1$$

$$=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\int_1^3 \frac{1}{x^2+x+1}\,dx=\int_1^3 \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

Let $u=x+\frac{1}{2}\Rightarrow du=dx$

When $x=1$, $u=\frac{3}{2}$

When $x=3$, $u=\frac{7}{2}$

$$\int_1^3 \frac{1}{x^2+x+1}\,dx=\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$$

$$=\frac{1}{\left(\sqrt{3}/2\right)}\left[\arctan\left(\frac{u}{\sqrt{3}/2}\right)\right]_{\frac{3}{2}}^{\frac{7}{2}}$$

$$=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2u}{\sqrt{3}}\right)\right]_{\frac{3}{2}}^{\frac{7}{2}}$$

$$=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{7}{\sqrt{3}}\right)-\arctan\left(\frac{3}{\sqrt{3}}\right)\right]$$

$$=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{7}{\sqrt{3}}\right)-\arctan\left(\sqrt{3}\right)\right]$$

$$=0.325\text{ (3 s.f.)}$$

3 c $\displaystyle\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx$

$$2+3x-2x^2 = 2-(2x^2-3x)$$
$$= 2-2\left(x^2-\frac{3}{2}x\right)$$
$$= 2-2\left[\left(x-\frac{3}{4}\right)^2-\frac{9}{16}\right]$$
$$= 2-2\left[\left(x-\frac{3}{4}\right)^2\right]+\frac{18}{16}$$
$$= \frac{50}{16}-2\left[\left(x-\frac{3}{4}\right)^2\right]$$
$$= 2\left[\frac{25}{16}-\left(x-\frac{3}{4}\right)^2\right]$$

$$\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{2\left[\frac{25}{16}-\left(x-\frac{3}{4}\right)^2\right]}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int_0^1 \frac{1}{\sqrt{\left(\frac{5}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}}\,dx$$

Let $u = x-\frac{3}{4} \Rightarrow du = dx$

When $x=0$, $u=-\frac{3}{4}$

When $x=1$, $u=\frac{1}{4}$

$$\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int_{-\frac{3}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{\left(\frac{5}{4}\right)^2-u^2}}\,du$$
$$= \frac{1}{\sqrt{2}}\left[\arcsin\left(\frac{u}{5/4}\right)\right]_{-\frac{3}{4}}^{\frac{1}{4}}$$
$$= \frac{1}{\sqrt{2}}\left[\arcsin\left(\frac{4u}{5}\right)\right]_{-\frac{3}{4}}^{\frac{1}{4}}$$
$$= \frac{1}{\sqrt{2}}\left[\arcsin\left(\frac{1}{5}\right)-\arcsin\left(-\frac{3}{5}\right)\right]$$
$$= 0.597 \text{ (3 s.f.)}$$

4 a $\displaystyle\int_1^3 \frac{1}{\sqrt{x^2-2x+2}}\,dx = \int_1^3 \frac{1}{\sqrt{(x-1)^2+1}}\,dx$

Let $u = x-1 \Rightarrow du = dx$

When $x = 1$, $u = 0$

When $x = 3$, $u = 2$

$$\int_1^3 \frac{1}{\sqrt{x^2-2x+2}}\,dx = \int_0^2 \frac{1}{\sqrt{u^2+1}}\,du$$

$$= [\text{arsinh}\, u]_0^2$$

$$= \left[\ln\left(x+\sqrt{x^2+1}\right)\right]_0^2$$

$$= \ln\left(2+\sqrt{2^2+1}\right) - \ln(1)$$

$$= \ln\left(2+\sqrt{5}\right) - \ln(1)$$

$$= \ln\left(2+\sqrt{5}\right)$$

4 b $\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx$

$$1+6x-3x^2 = 3\left(\frac{1}{3}+2x-x^2\right)$$
$$= 3\left(\frac{1}{3}+1-(1-x)^2\right)$$
$$= 3\left(\frac{4}{3}-(1-x)^2\right)$$

$$\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx = \int_1^2 \frac{1}{\sqrt{3\left(\frac{4}{3}-(1-x)^2\right)}}\,dx$$
$$= \frac{1}{\sqrt{3}}\int_1^2 \frac{1}{\sqrt{\frac{4}{3}-(1-x)^2}}\,dx$$

Let $u = 1-x \Rightarrow dx = -du$
When $x = 1$, $u = 0$
When $x = 2$, $u = -1$

$$\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx = -\frac{1}{\sqrt{3}}\int_0^{-1} \frac{1}{\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2-u^2}}\,du$$
$$= -\frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{u}{2/\sqrt{3}}\right)\right]_0^{-1}$$
$$= -\frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{\sqrt{3}u}{2}\right)\right]_0^{-1}$$
$$= -\frac{1}{\sqrt{3}}\arcsin\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{3\sqrt{3}}$$

5 $\int_1^3 \frac{1}{\sqrt{3x^2-6x+7}}\,dx$

$$3x^2-6x+7=3\left(x^2-2x+\frac{7}{3}\right)$$
$$=3\left((x-1)^2-1+\frac{7}{3}\right)$$
$$=3\left((x-1)^2+\frac{4}{3}\right)$$

$$\int_1^3 \frac{1}{\sqrt{3x^2-6x+7}}\,dx=\int_1^3 \frac{1}{\sqrt{3\left((x-1)^2+\frac{4}{3}\right)}}\,dx$$
$$=\frac{1}{\sqrt{3}}\int_1^3 \frac{1}{\sqrt{(x-1)^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,dx$$

Let $u=x-1 \Rightarrow dx=du$

When $x=1$, $u=0$

When $x=3$, $u=2$

$$\int_1^3 \frac{1}{\sqrt{3x^2-6x+7}}\,dx= =\frac{1}{\sqrt{3}}\int_0^2 \frac{1}{\sqrt{u^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,du$$
$$=\frac{1}{\sqrt{3}}\left[\operatorname{arsinh}\left(\frac{u}{2/\sqrt{3}}\right)\right]_0^2$$
$$=\frac{1}{\sqrt{3}}\left[\operatorname{arsinh}\left(\frac{\sqrt{3}u}{2}\right)\right]_0^2$$
$$=\frac{1}{\sqrt{3}}\left[\operatorname{arsinh}\left(\sqrt{3}\right)-\operatorname{arsinh}(0)\right]$$
$$=\frac{1}{\sqrt{3}}\left[\ln\left(\sqrt{3}+\sqrt{\sqrt{3}^2+1}\right)-0\right]$$
$$=\frac{1}{\sqrt{3}}\ln\left(2+\sqrt{3}\right)$$

6 a $\int \frac{1}{\sqrt{x^2+4x+5}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+1}}\,dx$

Let $u = x+2 \Rightarrow du = dx$

$\int \frac{1}{\sqrt{x^2+4x+5}}\,dx = \int \frac{1}{\sqrt{u^2+1}}\,du$

Let $u = \sinh\theta \Rightarrow du = \cosh\theta\,d\theta$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+4x+5}}\,dx &= \int \frac{1}{\sqrt{\sinh^2\theta+1}}\cosh\theta\,d\theta\\ &= \int \frac{1}{\sqrt{\cosh^2\theta}}\cosh\theta\,d\theta\\ &= \int d\theta\\ &= \theta + c\\ &= \text{arsinh}\,u + c\\ &= \text{arsinh}\,(x+2)+c\end{aligned}$$

6 b $\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx$

$$\begin{aligned}-x^2+4x+5 &= -(x-2)^2+4+5\\ &= 9-(x-2)^2\end{aligned}$$

$\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx = \int \frac{1}{\sqrt{9-(x-2)^2}}\,dx$

Let $u = x-2 \Rightarrow du = dx$

$\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx = \int \frac{1}{\sqrt{9-u^2}}\,du$

Let $u = 3\sin\theta \Rightarrow du = 3\cos\theta\,d\theta$

$$\begin{aligned}\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx &= \int \frac{1}{\sqrt{9-9\sin^2\theta}}\times 3\cos\theta\,d\theta\\ &= 3\int \frac{1}{\sqrt{9(1-\sin^2\theta)}}\times\cos\theta\,d\theta\\ &= \int \frac{1}{\sqrt{\cos^2\theta}}\times\cos\theta\,d\theta\\ &= \int d\theta\\ &= \theta+c\\ &= \arcsin\left(\frac{u}{3}\right)+c\\ &= \arcsin\left(\frac{x-2}{3}\right)+c\end{aligned}$$

7 $\displaystyle\int_{-0.2}^{0} \frac{1}{25x^2+10x+4}\,dx$

$$25x^2+10x+4=(5x+1)^2-1+4$$
$$=(5x+1)^2+3$$

$$\int_{-0.2}^{0} \frac{1}{25x^2+10x+4}\,dx=\int_{-0.2}^{0} \frac{1}{(5x+1)^2+3}\,dx$$

Let $x=\frac{1}{5}\left(\sqrt{3}\tan\theta-1\right)\Rightarrow dx=\frac{\sqrt{3}}{5}\sec^2\theta\,d\theta$

$$(5x+1)^2=\left(5\left[\frac{1}{5}\left(\sqrt{3}\tan\theta-1\right)\right]+1\right)^2$$
$$=\left(\sqrt{3}\tan\theta-1+1\right)^2$$
$$=3\tan^2\theta$$

When $x=0$, $x=\frac{1}{5}\left(\sqrt{3}\tan\theta-1\right)\Rightarrow\tan\theta=\frac{1}{\sqrt{3}}\Rightarrow\theta=\frac{\pi}{6}$

When $x=-0.2$, $x=\frac{1}{5}\left(\sqrt{3}\tan\theta-1\right)\Rightarrow\tan\theta=0\Rightarrow\theta=0$

$$\int_{-0.2}^{0} \frac{1}{25x^2+10x+4}\,dx=\int_{0}^{\frac{\pi}{6}} \frac{1}{3\tan^2\theta+3}\times\frac{\sqrt{3}}{5}\sec^2\theta\,d\theta$$
$$=\frac{\sqrt{3}}{5}\int_{0}^{\frac{\pi}{6}} \frac{1}{3\left(\tan^2\theta+1\right)}\sec^2\theta\,d\theta$$
$$=\frac{\sqrt{3}}{5}\int_{0}^{\frac{\pi}{6}} \frac{1}{3\sec^2\theta}\sec^2\theta\,d\theta$$
$$=\frac{\sqrt{3}}{15}\int_{0}^{\frac{\pi}{6}} d\theta$$
$$=\frac{\sqrt{3}}{15}\left[\frac{\pi}{6}-0\right]$$
$$=\frac{\pi\sqrt{3}}{90}$$

8 $\displaystyle\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,dx$

$$(x-2)(x+4)=x^2+2x-8$$
$$=(x+1)^2-1-8$$
$$=(x+1)^2-9$$

$$\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,dx=\int_3^4 \frac{1}{\sqrt{(x+1)^2-9}}\,dx$$

Let $u=x+1 \Rightarrow du=dx$
When $x=3$, $u=4$
When $x=4$, $u=5$

$$\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,dx=\int_4^5 \frac{1}{\sqrt{u^2-3^2}}\,du$$
$$=\left[\text{arcosh}\left(\frac{u}{3}\right)\right]_4^5$$
$$=\left[\ln\left(\frac{u}{3}\right)+\sqrt{\frac{u^2}{9}-1}\right]_4^5$$
$$=\ln\left(\frac{5}{3}+\sqrt{\frac{25}{9}-1}\right)-\ln\left(\frac{4}{3}+\sqrt{\frac{16}{9}-1}\right)$$
$$=\ln\left(\frac{5}{3}+\sqrt{\frac{16}{9}}\right)-\ln\left(\frac{4}{3}+\sqrt{\frac{7}{9}}\right)$$
$$=\ln\left(\frac{5}{3}+\frac{4}{3}\right)-\ln\left(\frac{4}{3}+\frac{\sqrt{7}}{3}\right)$$
$$=\ln(3)-\ln\left(\frac{4+\sqrt{7}}{3}\right)$$
$$=\ln\left(\frac{9}{4+\sqrt{7}}\right)$$
$$=\ln\left(\frac{9\left(4-\sqrt{7}\right)}{\left(4+\sqrt{7}\right)\left(4-\sqrt{7}\right)}\right)$$
$$=\ln\left(\frac{9\left(4-\sqrt{7}\right)}{16-7}\right)$$
$$=\ln\left(4-\sqrt{7}\right)$$

9 $\int \frac{1}{\left(x^2-2x+2\right)^{\frac{3}{2}}}\,dx$

$$x^2-2x+2=(x-1)^2-1+2$$
$$=(x-1)^2+1$$

$$\int \frac{1}{\left(x^2-2x+2\right)^{\frac{3}{2}}}\,dx=\int \frac{1}{\left((x-1)^2+1\right)^{\frac{3}{2}}}\,dx$$

Let $x=1+\sinh\theta \Rightarrow dx=\cosh\theta\,d\theta$

$$\int \frac{1}{\left(x^2-2x+2\right)^{\frac{3}{2}}}\,dx=\int \frac{1}{\left(\left((1+\sinh\theta)-1\right)^2+1\right)^{\frac{3}{2}}}\times\cosh\theta\,d\theta$$
$$=\int \frac{\cosh\theta}{\left(\sinh^2\theta+1\right)^{\frac{3}{2}}}\,d\theta$$
$$=\int \frac{\cosh\theta}{\left(\cosh^2\theta\right)^{\frac{3}{2}}}\,d\theta$$
$$=\int \frac{\cosh\theta}{\cosh^3\theta}\,d\theta$$
$$=\int \frac{1}{\cosh^2\theta}\,d\theta$$
$$=\int \operatorname{sech}^2\theta\,d\theta$$
$$=\tanh\theta+c$$
$$=\frac{\sinh\theta}{\cosh\theta}+c$$
$$=\frac{\sinh\theta}{\sqrt{1+\sinh^2\theta}}+c$$
$$=\frac{x-1}{\sqrt{1+(x-1)^2}}+c$$
$$=\frac{x-1}{\sqrt{x^2-2x+1}}+c$$

10 $\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$

Let $x = 2\sin\theta - 1 \Rightarrow \mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x = \int \frac{2\sin\theta - 1}{\sqrt{3-2(2\sin\theta-1)-(2\sin\theta-1)^2}} \times 2\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{2\cos\theta(2\sin\theta-1)}{\sqrt{5-4\sin\theta-(4\sin^2\theta-4\sin\theta+1)}}\,\mathrm{d}\theta$$

$$= \int \frac{2\cos\theta(2\sin\theta-1)}{\sqrt{4-4\sin^2\theta}}\,\mathrm{d}\theta$$

$$= \int \frac{2\cos\theta(2\sin\theta-1)}{\sqrt{4(1-\sin^2\theta)}}\,\mathrm{d}\theta$$

$$= \int \frac{\cos\theta(2\sin\theta-1)}{\sqrt{\cos^2\theta}}\,\mathrm{d}\theta$$

$$= \int (2\sin\theta - 1)\,\mathrm{d}\theta$$

$$= -2\cos\theta - \theta + c$$

$$= -2\sqrt{1-\sin^2\theta} - \theta + c$$

$$x = 2\sin\theta - 1 \Rightarrow \sin\theta = \frac{x+1}{2} \Rightarrow \theta = \arcsin\left(\frac{x+1}{2}\right)$$

Therefore:

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x = -2\sqrt{1-\left(\frac{x+1}{2}\right)^2} - \arcsin\left(\frac{x+1}{2}\right) + c$$

$$= -\sqrt{4-(x^2+2x+1)} - \arcsin\left(\frac{x+1}{2}\right) + c$$

$$= -\sqrt{3-2x-x^2} - \arcsin\left(\frac{x+1}{2}\right) + c$$

Challenge

1 a $\int x\cosh^2(x^2)\,dx$

Let $u = x^2 \Rightarrow du = 2x\,dx$

$$\int x\cosh^2(x^2)\,dx = \frac{1}{2}\int \cosh^2 u\,du$$
$$= \frac{1}{2}\int \frac{\cosh 2u + 1}{2}\,du$$
$$= \frac{1}{4}\int (\cosh 2u + 1)\,du$$
$$= \frac{1}{4}\left(\frac{1}{2}\sinh 2u + u\right) + c$$
$$= \frac{1}{8}\sinh 2u + \frac{1}{4}u + c$$
$$= \frac{1}{8}\sinh(2x^2) + \frac{1}{4}x^2 + c$$

b $\int \frac{x}{\cosh^2(x^2)}\,dx$

Let $u = x^2 \Rightarrow du = 2x\,dx$

$$\int \frac{x}{\cosh^2(x^2)}\,dx = \frac{1}{2}\int \frac{1}{\cosh^2 u}\,du$$
$$= \frac{1}{2}\int \operatorname{sech}^2 u\,du$$
$$= \frac{1}{2}\tanh u + c$$
$$= \frac{1}{2}\tanh(x^2) + c$$